Spinning Superconducting Semilocal Strings

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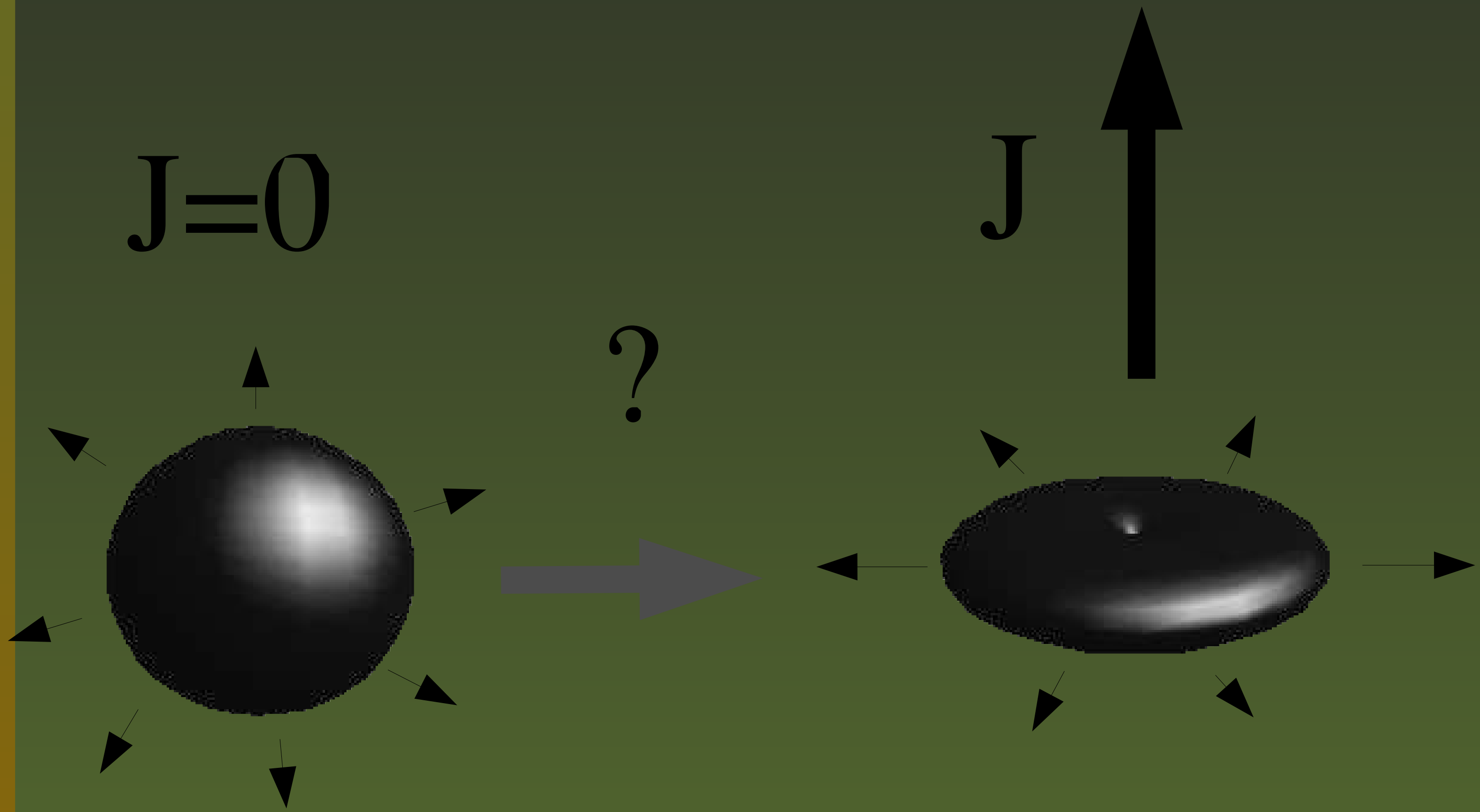
Search of spinning solitons

- Solitons are solutions of non-linear (exact/effective) field equations describing localized, particle-like objects – lumps of energy monopoles, dyons, vortices, sphalerons, skyrmions, knots, Q-balls ...

- Almost all known soliton solution have vanishing angular momentum, $J = 0$

Do static solitons

admit *stationary*, spinning generalizations ?



Results

- <u>Global solitons DO rotate</u> – in theories with rigid symmetries there are stationary, spinning solitons spinning Q-balls, global vortices (in helium ...) . Their energy is generically infinite.

- <u>Local solitons DO NOT rotate</u> – none of the known gauge field Yang-Mills-Higgs solitons with gauge group $G \leq SU(2)$ (t'Hooft-Polyakov monopoles, Julia-Zee dyons, sphalerons, ANO-vortices) admit spinning generalisations within the manifestly stationary and axially symmetric sector.

Let us try a mixture: local+global

Semilocal Abelian Higgs Model

$$L = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (D_\mu\Phi)^\dagger D^\mu\Phi - \frac{\beta}{4}(\Phi^\dagger\Phi - 1)^2 .$$

$$\Phi = \left(\begin{array}{c} \Phi^+ \\ \Phi^- \end{array}\right), \quad D_\mu\Phi = (\partial_\mu + iA_\mu)\Phi.$$

$\Theta_W \to 0$ limit of Standard Model:

$$U(1)_{\mathrm{local}} \times SU(2)_{\mathrm{local}} \to U(1)_{\mathrm{local}} \times SU(2)_{\mathrm{global}}$$

Four conserved Noether currents

$$j^\mu_B = i\{(D^\mu\Phi)^\dagger M_B\Phi - \Phi^\dagger M_B D^\mu\Phi\} \tag{1}$$

$M_0 = \mathbf{1}$, $M_a = \tau_a$

Field equations

$$\nabla_\mu F^{\mu\nu} = j_0^\mu,$$
$$D_\mu D^\mu \Phi = -\beta(\Phi^\dagger\Phi - 1)\Phi.$$

Known solitons /Achucarro, Vachaspati, 2000/

- $\beta > 0$: A_μ, $\begin{pmatrix} \Phi^+ \\ 0 \end{pmatrix}$ – ANO vortices
- $\beta = \left(\frac{M_H}{M_W}\right)^2 = 1$: A_μ, $\begin{pmatrix} \Phi^+ \\ \Phi^- \end{pmatrix}$ – BPS skyrmions

Are there other solutions ?

Cylindrical symmetry

Action is invariant under the symmetry generated by

$$K_{(1)} = \frac{\partial}{\partial t}, \quad K_{(2)} = \frac{\partial}{\partial z}, \quad K_{(3)} = \frac{\partial}{\partial \varphi}.$$

Conserved currents $j^{\mu}_{(m)} = T^{\mu}_{\nu} K^{\nu}_{(m)}$, with

$$T^{\mu}_{\nu} = -F^{\mu\sigma} F_{\nu\sigma} + (D^{\mu}\Phi)^{\dagger} D_{\nu}\Phi + (D_{\nu}\Phi)^{\dagger} D^{\mu}\Phi - \delta^{\mu}_{\nu} L$$

the energy-momentum tensor.

Conserved charges

$$E = 2\pi \int_0^\infty \rho\, T_0^0\, d\rho \quad \text{energy}$$

$$P = 2\pi \int_0^\infty \rho\, T_z^0\, d\rho \quad \text{momentum}$$

$$J = 2\pi \int_0^\infty \rho T_\varphi^0\, d\rho \quad \text{angular momentum}$$

In addition, one has four $U(1) \times SU(2)$ charges.

$$\mathcal{Q}_A = 2\pi \int_0^\infty \rho\, j_A^0\, d\rho$$

$A = 0, 1, 2, 3$

Symmetry conditions

Under the action of $K_{(m)}$ the fields are invariant
up to a phase rotation, local or global:

$$K_{(m)} A_\mu = -\partial_\mu \alpha_{(m)}(x),$$
$$K_{(m)} \Phi = i(\alpha_{(m)}(x) + \theta^a_{(m)} \tau_a)\Phi,$$

The symmetry is *manifest* if $\exists$ (locally) a gauge where the r.h.s. vanish – nothing depends on t, z, φ. Otherwise, symmetry is *non-manifest*

Solving the symmetry conditions

$K_{(m)}$ commute $\Rightarrow \exists$ gauge where $A_\mu = A_\mu(\rho)$,

$$K_{(m)}\,\Phi = i(\alpha_{(m)} + \theta_{(m)}\tau_3)\Phi,$$

whose solution

$$\Phi^\pm = F^\pm(\rho)\exp\{\pm i(\alpha_{(m)} \pm \theta_{(m)})x^m\}$$

$x^m \equiv (t, z, \varphi)$. Residual gauge freedom$\Rightarrow$ one can set

$$A_\rho = 0, \quad \alpha_{(z)} + \theta_{(z)} = 0,$$

$$\alpha_{(t)} - \theta_{(t)} = 0, \;\; \alpha_{(\varphi)} - \theta_{(\varphi)} = 0$$

Symmetric fields

$$A_\mu dx^\mu = (\Omega(\rho) - \omega)dt - P(\rho)dz + (W(\rho) - \nu)d\varphi$$

$$\Phi = \begin{pmatrix} f(\rho)e^{i(U(\rho)+\omega t+\nu\varphi)} \\ g(\rho)e^{i(V(\rho)+kz)} \end{pmatrix}$$

with $\omega, k \in R$, $\nu \in Z$, and Ω, P, U, V, W, f, g – seven functions of ρ. Symmetry is non-manifest.

Further reduction is imposed by finiteness of energy

Regularity conditions

$$(\rho f^2 U')' = 0 \;\Rightarrow\; U' = \frac{C_u}{\rho f^2},$$

$\Rightarrow$ energy is divergent unless $C_u = 0 \Rightarrow U = const.$ Same for V.

$$\xi = k\Omega - \omega P.$$

satisfies

$$\rho\xi'' + \xi' = 2\rho(f^2 + g^2)\xi$$

$\Rightarrow$ energy is divergent unless $\xi(\rho) = 0 \Rightarrow$

$$\Omega(\rho) = \omega Y(\rho), \qquad P(\rho) = kY(\rho).$$

Most general ansatz

$$A_\mu dx^\mu = \omega(Y-1)dt - kY dz + (W-\nu)d\varphi$$

$$\Phi = \begin{pmatrix} f e^{i(\omega t+\nu\varphi)} \\ g e^{ikz)} \end{pmatrix}$$

where ω, k are real constants, the winding number ν is integer, and Y, W, f, g – four functions of ρ.

Boundary conditions for $0 \leftarrow \rho \rightarrow \infty$

$$0 \leftarrow f(\rho) \rightarrow 1,$$

$$\nu \leftarrow W(\rho) \rightarrow 0,$$

$$g(0), Y(0) \leftarrow g(\rho), Y(\rho) \rightarrow 0.$$

Boost symmetry

After $x^\mu = (t, z, \varphi, \rho) \to \tilde{x}'^\mu = (\tilde{t}, \tilde{z}, \varphi, \rho)$, with

$$t = \tilde{t}\cosh(\gamma) - \tilde{z}\sinh(\gamma), \qquad z = \tilde{z}\cosh(\gamma) - \tilde{t}\sinh(\gamma),$$

and the gauge transformation

$$A_\mu \to A_\mu - \partial_\mu \alpha(x), \qquad \Phi \to e^{i\alpha(x)}\Phi,$$

with $\alpha = (\omega\tilde{z} + k\tilde{t})\sinh(\gamma)$, the ansatz restores its form up to

$$\tilde{\omega} = \cosh(\gamma)\omega + \sinh(\gamma)k, \qquad \tilde{k} = \cosh(\gamma)k + \sinh(\gamma)\omega.$$

$\Rightarrow$ all values (ω, k) belonging to the same orbit with fixed $\sigma^2 = k^2 - \omega^2$ are equivalent

Lorentz invariants

do not depend separately on ω, k but on

$$\sigma^2 = k^2 - \omega^2$$

Example – the reduced Lagrangian

$$
\begin{aligned}
-\rho L &= \sigma^2 \rho \left(\frac{1}{2} Y'^2 + f^2 Y^2 + g^2 (Y-1)^2 \right) \\
&+ \rho (f'^2 + g'^2) + \frac{1}{2\rho} W'^2 + \frac{1}{\rho} f^2 W^2 + \frac{1}{\rho} g^2 (W-\nu)^2 \\
&+ \frac{\beta\rho}{2} (f^2 + g^2 - 1)^2
\end{aligned}
$$

Field equations

$$
\begin{aligned}
\frac{1}{\rho}(\rho Y')' &= 2(f^2+g^2)Y-2g^2, \\
\rho\left(\frac{W'}{\rho}\right)' &= 2f^2W+2g^2(W-\nu), \\
\frac{1}{\rho}(\rho f')' &= \{\sigma^2Y^2+\frac{W^2}{\rho^2}+\beta(f^2+g^2-1)\}f, \\
\frac{1}{\rho}(\rho g')' &= \{\sigma^2(Y-1)^2+\frac{(W-\nu)^2}{\rho^2} \\
&+ \beta(f^2+g^2-1)\}g.
\end{aligned}
$$

Charges

$$2\mathcal{Q} = 2\pi \int_0^\infty \rho j_3^0 d\rho = 8\pi\omega \int_0^\infty \rho g^2 (1-Y) d\rho,$$

superconducting current along the vortex

$$\mathcal{I} = 2\pi \int_0^\infty \rho j_3^z d\rho = 2\,\frac{k}{\omega}\,\mathcal{Q},$$

momentum and angular momentum

$$\begin{aligned} P &= 2\pi \int_0^\infty \rho\, T_z^0 d\rho = -k\mathcal{Q}, \\ J &= 2\pi \int_0^\infty \rho\, T_\varphi^0 d\rho = \nu\mathcal{Q}. \end{aligned}$$

Energy

$$
\begin{aligned}
E &= 2\pi|\nu| + \frac{\omega^2+k^2}{2\omega}\,\mathcal{Q} \\
&+2\pi\,\frac{\beta-1}{2}\int_0^\infty d\rho\rho\,(f^2+g^2-1)^2 \\
+ &\;\; 2\pi\int_0^\infty \frac{d\rho}{\rho}\Big\{(\rho f'-fW)^2+(\rho g'-g(W-\nu))^2 \\
&+\frac{1}{2}(W'-\rho(f^2+g^2-1))^2\Big\} \\
&\geq 2\pi|\nu| + \frac{\omega^2+k^2}{2\omega}\,\mathcal{Q} \qquad \text{for } \beta>1
\end{aligned}
$$

Three types of solutions

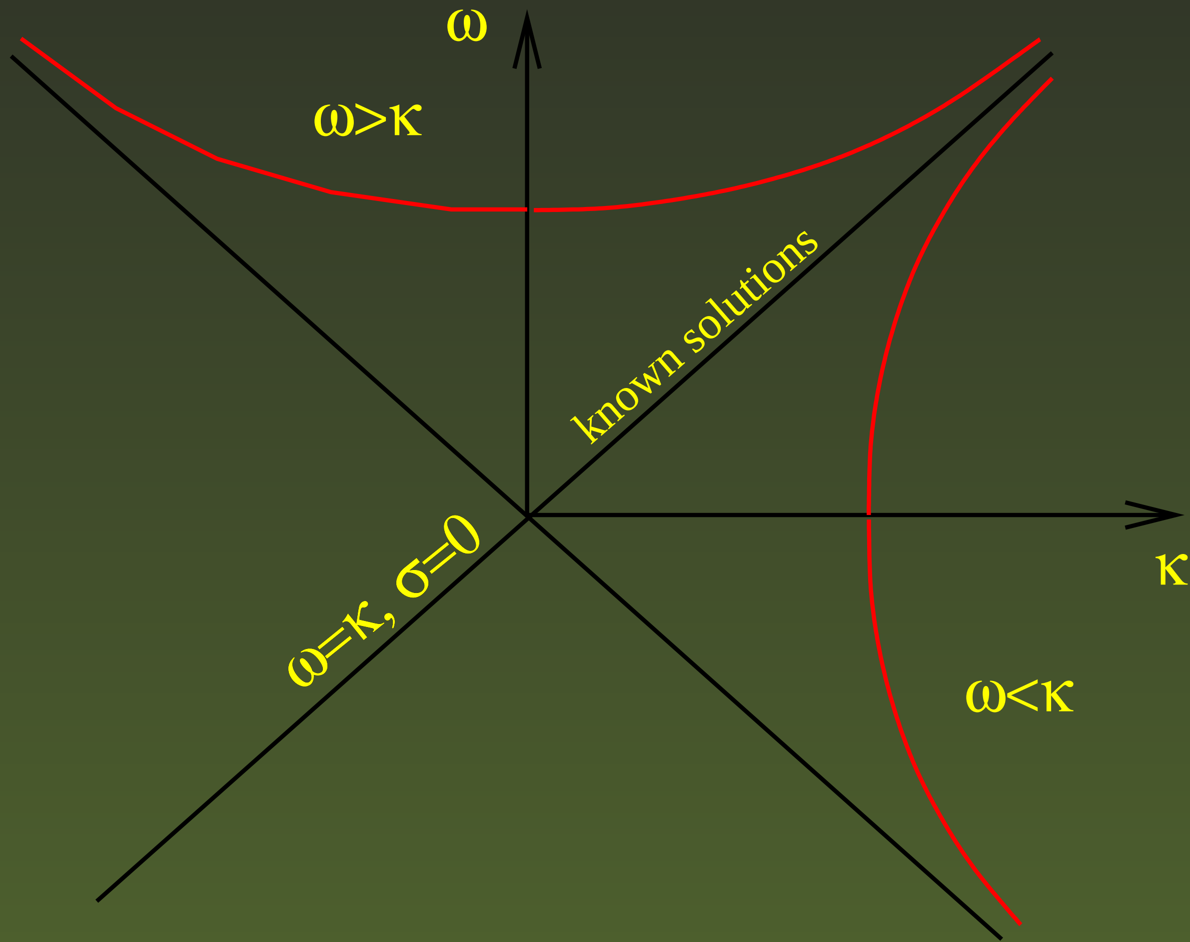
All (ω, k) on the same Lorentz orbit are equivalent.
Three types of orbits:

- $\sigma^2 = 0$ Chiral case, contains $(\omega, k) = 0$. All known solutions are of this type.

- $\sigma^2 < 0$ Timelike case – no finite energy solutions.

- $\sigma^2 > 0$ Spacelike case – new spinning strings.
  There exists a frame where $\omega = 0$, $\sigma^2 = k^2$, $\mathcal{Q} = P = J = 0$ – charges are not Lorentz invariant !

Lorentz orbits

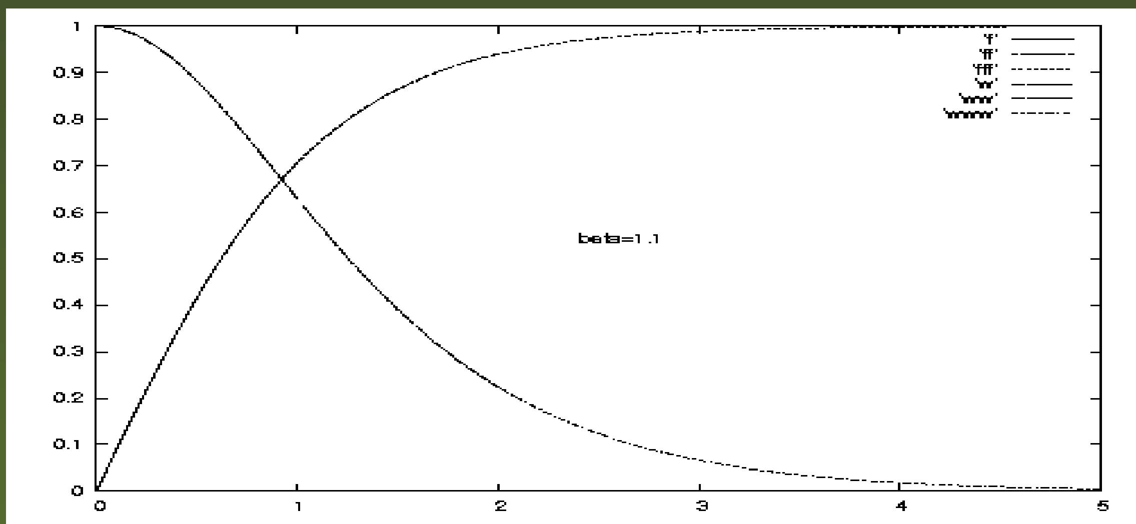


Consider the known solutions with $\sigma^2 = 0$:

$\sigma = 0$, $g = 0 \Rightarrow Y = 0 \Rightarrow$ **ANO**

$$
\begin{aligned}
\rho\left(\frac{W'}{\rho}\right)' &= 2f^2W, \\
\frac{1}{\rho}(\rho f')' &= \left\{\frac{W^2}{\rho^2} + \beta(f^2-1)\right\}f,
\end{aligned}
$$

$$\mathcal{Q} = \mathcal{I} = P = J = 0$$



$\sigma = 0$, $g \neq 0$

$$\frac{1}{\rho}(\rho Y')' = 2(f^2 + g^2)Y - 2g^2$$

$$\rho\left(\frac{W'}{\rho}\right)' = 2f^2 W + 2g^2(W - \nu),$$

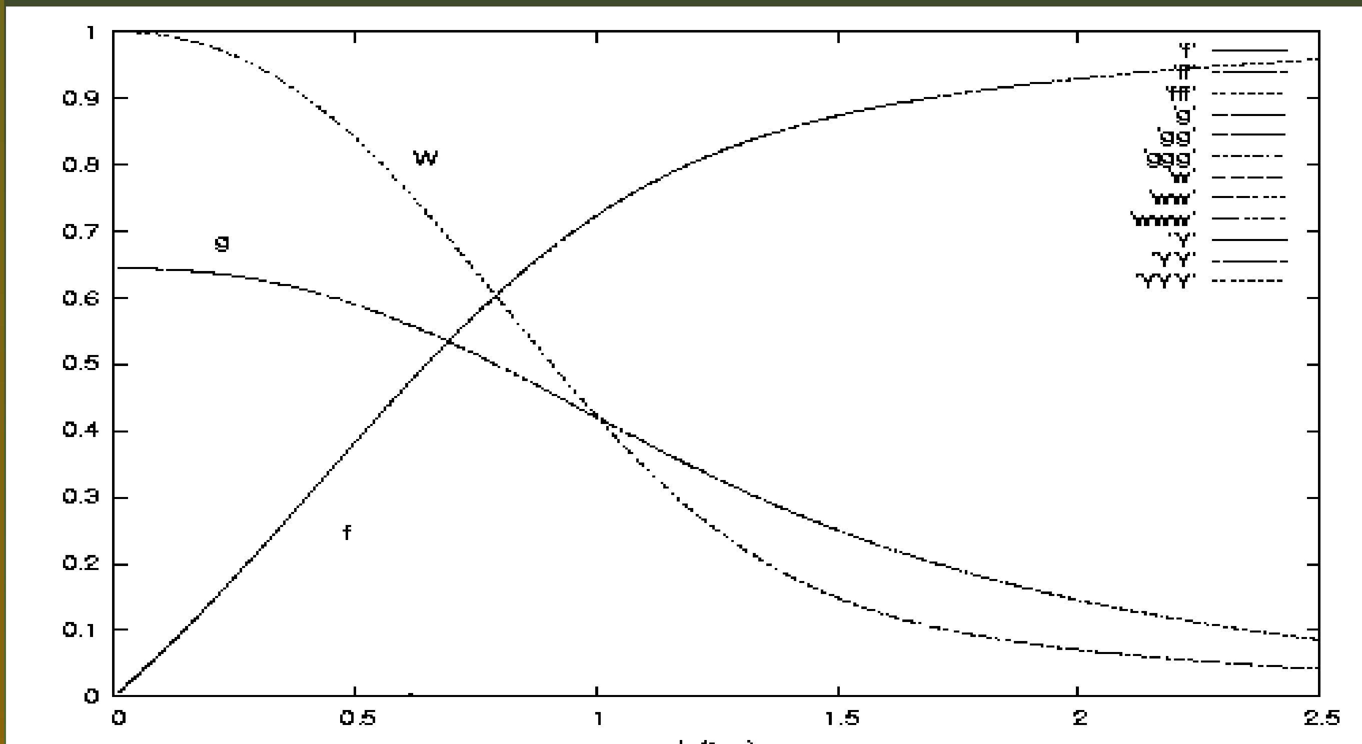
$$\frac{1}{\rho}(\rho f')' = \{\frac{W^2}{\rho^2} + \beta(f^2 + g^2 - 1)\}f,$$

$$\frac{1}{\rho}(\rho g')' = \{\frac{(W-\nu)^2}{\rho^2} + \beta(f^2 + g^2 - 1)\}g$$

Solutions are known for $\beta = 1$
/Achucarro, Vachaspati ’91/, /Hindnarsh ’92/

BPS equations ⇒ ‘skyrmions’

$$\rho f' = fW, \qquad g = (C/\rho)^\nu f$$
$$W' = \rho\left(f^2 + \left(\frac{C}{\rho}\right)^\nu f^2 - 1\right)$$



Skyrmions comprise a family labelled by $g(0)$.

- For $\omega = k = 0$ they are static, $\mathcal{Q} = \mathcal{I} = P = J = 0$, the energy attains the Bogomol'nyi bound

$$E = 2\pi|\nu|$$

- For $\omega = k \neq 0$ they spin,

$$\mathcal{I} = \pm\mathcal{Q}, \quad P = -k\mathcal{Q}, \quad J = \nu\mathcal{Q}$$

the energy

$$E = 2\pi|\nu| + |\omega|\mathcal{Q}$$

If $\nu = 1$, then $\mathcal{Q} = \infty$. /Abrahams '93/

New solutions

Skyrmions comprise a family parameterized by $q = g(0)$, known only for $\beta = 1$. Can we generalize them for any β as solutions of the minimal system with $\sigma = 0$?

$$
\begin{aligned}
\rho \left(\frac{W'}{\rho}\right)' &= 2f^2 W + 2g^2 (W - \nu), \\
\frac{1}{\rho}(\rho f')' &= \{\frac{W^2}{\rho^2} + \beta(f^2 + g^2 - 1)\} f, \\
\frac{1}{\rho}(\rho g')' &= \{\frac{(W-\nu)^2}{\rho^2} + \beta(f^2 + g^2 - 1)\} g
\end{aligned}
$$

Asymptotic solutions, $\sigma = 0$

One has for $0 \leftarrow \rho \rightarrow \infty$

$$1 - b\rho^2 + \ldots \leftarrow W \rightarrow \frac{Q^2}{\rho^2}(1+\ldots) + A\sqrt{\rho}\, e^{-\sqrt{2}\rho}(1+\ldots)$$

$$a\rho + \ldots \leftarrow f = 1 - \frac{Q^2}{2\rho^2}(1+\ldots) + \frac{B}{\sqrt{\rho}}\, e^{-\sqrt{2\beta}\rho}(1+\ldots)$$

$$q + O(\rho^2) \leftarrow g = \frac{Q}{\rho}\,(1 + O(\rho^{-1}))$$

$a, b, Q, A, B + q$ = 5 free + 1 fixed parameter $\Rightarrow$
not enough to fulfil 6 matching conditions $\Rightarrow$
no solutions for arbitrary β. What to do ?

Asymptotic solutions, $\sigma \neq 0$

$$p + O(\rho^2) \leftarrow Y = \frac{D}{\sqrt{\rho}} e^{-\sqrt{2}\rho}(1 + \ldots)$$

$$1 - b\rho^2 + \ldots \leftarrow W \rightarrow A\sqrt{\rho}\, e^{-\sqrt{2}\rho}(1 + \ldots)$$

$$a\rho + \ldots \leftarrow f = 1 + \frac{B}{\sqrt{\rho}} e^{-\sqrt{2\beta}\rho}(1 + \ldots)$$

$$q + O(\rho^2) \leftarrow g = \frac{C}{\sqrt{\rho}} e^{-\sigma\rho}(1 + \ldots)$$

a, b, Q, A, B, C, D+q = 7 free + 1 fixed parameter + free σ ⇒ *just enough* to fulfil 8 matching conditions ⇒ non-linear eigenvalue problem for σ^2.

Non-linear eigenvalue problem for σ^2

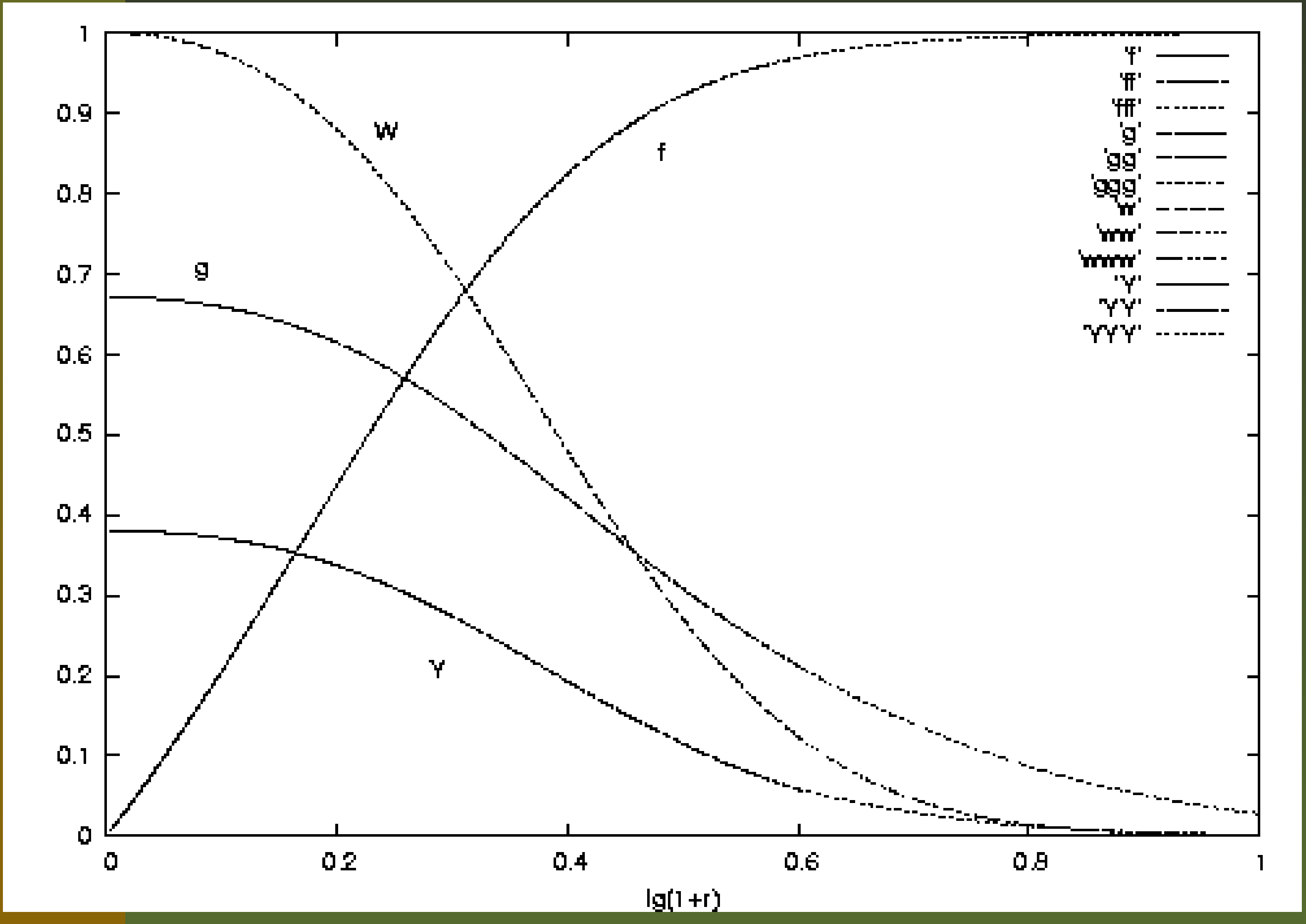
$$\begin{aligned}
\frac{1}{\rho}(\rho Y')' &= 2(f^2+g^2)Y-2g^2, \\
\rho\left(\frac{W'}{\rho}\right)' &= 2f^2W+2g^2(W-\nu), \\
\frac{1}{\rho}(\rho f')' &= \{\sigma^2Y^2+\frac{W^2}{\rho^2}+\beta(f^2+g^2-1)\}f,
\end{aligned}$$

$$\frac{1}{\rho}(\rho g')'=\{\sigma^2(Y-1)^2+\frac{(W-\nu)^2}{\rho^2}+\beta(f^2+g^2-1)\}g.$$

Solution exists for any $\beta, g(0)=q$.

Spinning strings

Exist for any $g(0)$ and $\beta \geq 1$



Some properties

Generalizations of $\beta=1$ skyrmions for $\beta>1$. Carry a conserved charge $\mathcal{Q}$ and the superconducting current $\mathcal{I}=2(k/\omega)\,\mathcal{Q}$, also carry the momentum and angular momentum

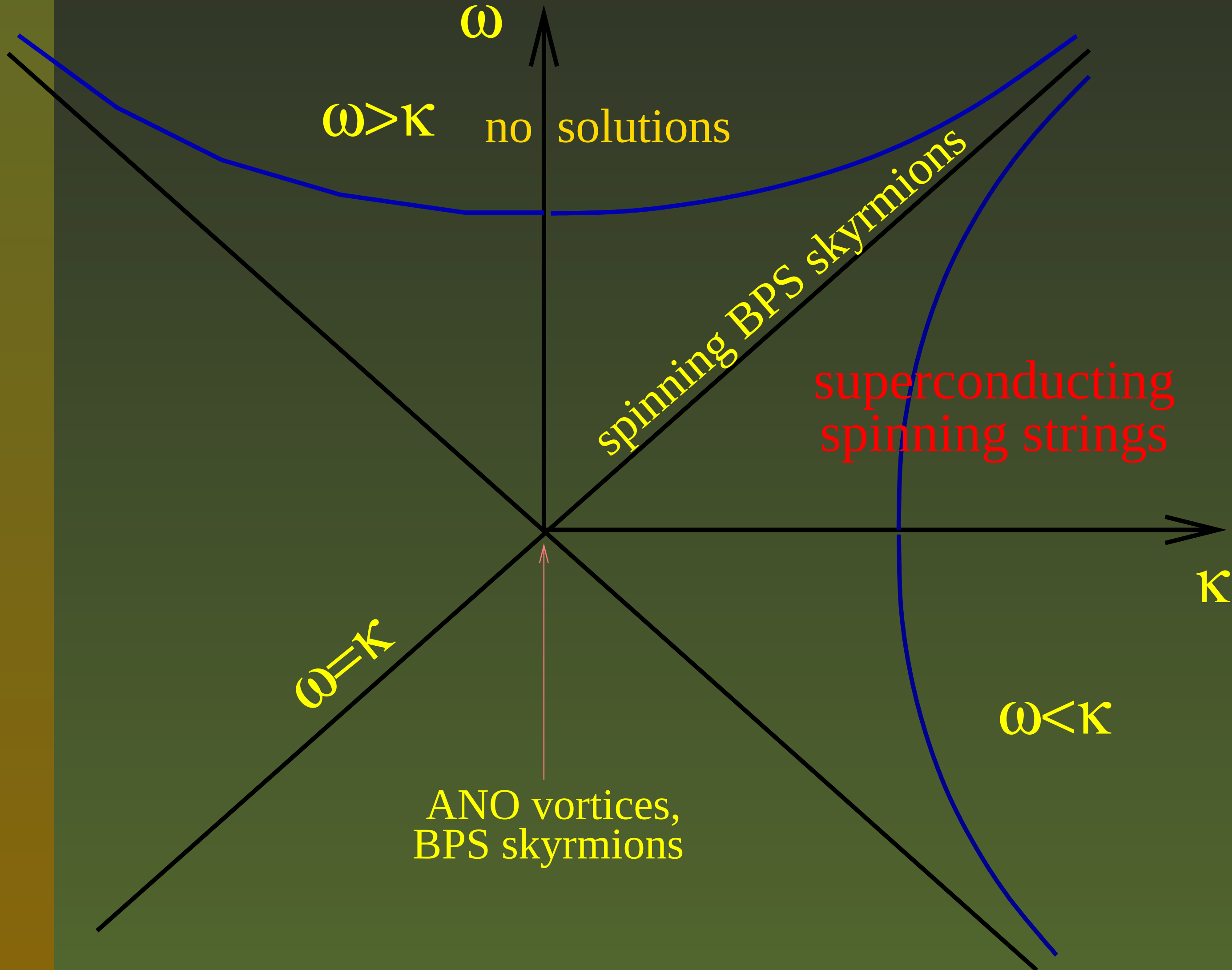
$$P=-k\mathcal{Q}\qquad J=\nu\mathcal{Q}$$

In the $\omega=0$ frame $\mathcal{Q}=P=J=0$, but $\mathcal{I}\neq 0$, the energy in this frame

$$E_{ANO}(\beta)\geq E\geq 2\pi|\nu|+\sigma\mathcal{Q}$$

Probably stable – have smaller energy than the ANO vortices. Probably can be promoted to solutions of standard model. Probably can be applied in type II superconductivity.

Summary



ω
ω>κ
no solutions
spinning BPS skyrmions
superconducting
spinning strings
κ
ω=κ
ω<κ
ANO vortices,
BPS skyrmions